السنة 2 بكالوريا علوم رياضية	الأعداد العقدية حلول مقترحة	سلسلة 3

تمرين 1 : $w=\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)$ ، $\begin{cases} A=1+\cos\left(\frac{\pi}{5}\right)+\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{3\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right) \\ B=\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{3\pi}{5}\right)+\sin\left(\frac{4\pi}{5}\right) \end{cases}$

1

$$A=1+\cos\left(\frac{\pi}{5}\right)+\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{3\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)$$

$$A=1+\cos\left(\frac{\pi}{5}\right)+\cos\left(\frac{2\pi}{5}\right)+\cos\left(\pi-\frac{2\pi}{5}\right)+\cos\left(\pi-\frac{\pi}{5}\right)$$

$$A=1+\cos\left(\frac{\pi}{5}\right)+\cos\left(\frac{2\pi}{5}\right)-\cos\left(\frac{\pi}{5}\right)-\cos\left(\frac{2\pi}{5}\right)$$

$$A=1$$

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$$A+iB=1+\left[\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right]+\left[\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)\right]+\left[\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)\right]\left[+\cos\left(\frac{4\pi}{5}\right)+i\sin\left(\frac{4\pi}{5}\right)\right]$$

$$A+iB=1+w+w^2+w^3+w^4=\frac{1-w^5}{1-w}$$

بما أن $w^5=\cos(\pi)+i\sin(\pi)=-1$ فإن : $A+iB=\frac{2}{1-w}$ منه : $iB=\frac{2}{1-w}-A=\frac{2}{1-w}-1=\frac{1+w}{1-w}$

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$$1-w=1-\cos\left(\frac{\pi}{5}\right)-i\sin\left(\frac{\pi}{5}\right)=2\sin^2\left(\frac{\pi}{10}\right)-2\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{10}\right)i=2\sin\left(\frac{\pi}{10}\right)\left(\sin\left(\frac{\pi}{10}\right)-\cos\left(\frac{\pi}{10}\right)i\right)$$

$$1-w=2\sin\left(\frac{\pi}{10}\right)\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)-\sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)i\right)=2\sin\left(\frac{\pi}{10}\right)\left(\cos\left(\frac{2\pi}{5}\right)-\sin\left(\frac{2\pi}{5}\right)i\right)$$

$$1-w=\left[2\sin\left(\frac{\pi}{10}\right);\frac{-2\pi}{5}\right]$$

$$1+w=1+\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)=2\cos^2\left(\frac{\pi}{10}\right)+2\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{10}\right)i=2\cos\left(\frac{\pi}{10}\right)\left(\cos\left(\frac{\pi}{10}\right)+\sin\left(\frac{\pi}{10}\right)i\right)$$

$$1+w=\left[2\cos\left(\frac{\pi}{10}\right);\frac{\pi}{10}\right]$$

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$$iB=\frac{\left[2\cos\left(\frac{\pi}{10}\right);\frac{\pi}{10}\right]}{\left[2\sin\left(\frac{\pi}{10}\right);\frac{-2\pi}{5}\right]}=\left[\frac{2\cos\left(\frac{\pi}{10}\right)}{2\sin\left(\frac{\pi}{10}\right)};\frac{\pi}{10}+\frac{2\pi}{5}\right]=\left[\text{cotan}\left(\frac{\pi}{10}\right);\frac{\pi}{2}\right]=i\ \text{cotan}\left(\frac{\pi}{10}\right)$$

بالتالي : $B=\text{cotan}\left(\frac{\pi}{10}\right)$

للتذكير ، $\text{cotan}(x)=\frac{\cos x}{\sin x}=\frac{1}{\tan x}$ (ليس من الضروري استعماله، لكنه يجعل الصيغة أكثر بساطة)

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$$B=\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{3\pi}{5}\right)+\sin\left(\frac{4\pi}{5}\right)$$

لدينا :
$$B=\cos\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\right)+\sin\left(\pi-\frac{2\pi}{5}\right)+\sin\left(\pi-\frac{\pi}{5}\right)$$

$$B=\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{\pi}{5}\right)+\sin\left(\frac{2\pi}{5}\right)$$

$$B=2C$$

بالتالي :
$$C=\frac{B}{2}=\frac{\operatorname{cotan}\left(\frac{\pi}{10}\right)}{2}$$

تمرين 2 : $w=\cos\left(\frac{\pi}{5}\right)+\boldsymbol{i}\sin\left(\frac{\pi}{5}\right)$

1

لدينا:
$$w^2+\frac{1}{w^2}-\left(w+\frac{1}{w}\right)+1=\frac{w^4+1-w^3-w+w^2}{w^2}=\frac{1+(-w)+(-w)^2+(-w)^3+(-w)^4}{w^2}$$

$$w^2+\frac{1}{w^2}-\left(w+\frac{1}{w}\right)+1=\frac{1}{w^2}\times\frac{1-(-w)^5}{1-(-w)}=\frac{1}{w^2}\times\frac{1+w^5}{1+w}$$

و بما أن $w^5=\cos(\pi)+\boldsymbol{i}\sin(\pi)=-1$ فإن : $w^2+\frac{1}{w^2}-\left(w+\frac{1}{w}\right)+1=0$

2

لدينا $z=w+\frac{1}{w}$ منه:

أ)
$$z^2-z-1=\left(w+\frac{1}{w}\right)^2-\left(w+\frac{1}{w}\right)-1=w^2+2+\frac{1}{w^2}-\left(w+\frac{1}{w}\right)-1=w^2+\frac{1}{w^2}-\left(w+\frac{1}{w}\right)+1=0$$

إذن z هو حل للمعادلة : $z^2-z-1=0$ ، $\Delta=1+4=5$ منه : $z=\frac{1+\sqrt{5}}{2}$ أو $z=\frac{1-\sqrt{5}}{2}$

ب) لدينا :
$$w+\frac{1}{w}=w+w^{-1}=\cos\left(\frac{\pi}{5}\right)+\boldsymbol{i}\sin\left(\frac{\pi}{5}\right)+\cos\left(\frac{-\pi}{5}\right)+\boldsymbol{i}\sin\left(\frac{-\pi}{5}\right)=2\cos\left(\frac{\pi}{5}\right)$$

3

من أ) و ب) نستنتج أن : أو $2\cos\left(\frac{\pi}{5}\right)=\frac{1-\sqrt{5}}{2}$

لكن وبما أن $\cos\left(\frac{\pi}{5}\right)>0$ (لأن $0<\frac{\pi}{5}<\frac{\pi}{2}$) فإن : $2\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{2}$ بالتالي : $\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$

منه :
$$\sin\left(\frac{\pi}{5}\right)=\sqrt{1-\cos^2\left(\frac{\pi}{5}\right)}=\sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2}=\sqrt{\frac{16-1-2\sqrt{5}-5}{16}}=\frac{\sqrt{10-2\sqrt{5}}}{4}=\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4}$$

و منه :
$$\tan\left(\frac{\pi}{5}\right)=\frac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)}=\sqrt{5-2\sqrt{5}}$$

- $\sin\left(\frac{\pi}{5}\right)=\sqrt{1-\cos^2\left(\frac{\pi}{5}\right)}$ لأن : $\sin\left(\frac{\pi}{5}\right)>0$
- تجاوزت بعض حسابات الجذور المربعة

تمرين 3 : المستوى العقدي منسوب إلى م.م.م $(O,\vec{e_1},\vec{e_2})$ ، ليكن $(n,\theta)\in IN^*\times IR$

لدينا لكل $k\in\{1;2...;n\}$:

$$z_k+1=1+\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)=2\cos^2\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)+2\sin\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)\sin\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)i$$

$$z_k+1=2\cos\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)\left(\cos\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)+\sin\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)i\right)$$

منه : $(z_k+1)^n=2^n\cos^n\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)\left(\cos\left(\frac{\theta}{2}+k\pi\right)+\sin\left(\frac{\theta}{2}+k\pi\right)i\right)$

طريقة 1

- إذا كان : $\cos^n\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)>0$ فإن : $\arg\left((z_k+1)^n\right)\equiv 0+\frac{\theta}{2}+k\pi\,[2\pi]\equiv\frac{\theta}{2}\,[\pi]$
- إذا كان : $\cos^n\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)<0$ فإن : $\arg\left((z_k+1)^n\right)\equiv \pi+\frac{\theta}{2}+k\pi\,[2\pi]\equiv\frac{\theta}{2}\,[\pi]$
- إذا كان : $\cos^n\left(\frac{\theta}{2n}+\frac{k\pi}{n}\right)=0$ فإن : $(z_k+1)^n=0$

في كل الحالات نجد أن M_k إما تنطبق مع O أو تنتمي للمستقيم المار من O و الذي يكون الزاوية $\frac{\theta}{2}$ مع $(O,\vec{e_1})$، بالتالي النقط M_1 و M_2 و ... و M_n مستقيمية

طريقة 2: لدينا لكل $p,q\in\{1;2...;n\}$

$$\frac{(z_p+1)^n}{(z_q+1)^n}=\frac{2^n\cos^n\left(\frac{\theta}{2n}+\frac{p\pi}{n}\right)\left(\cos\left(\frac{\theta}{2}+p\pi\right)+\sin\left(\frac{\theta}{2}+p\pi\right)i\right)}{2^n\cos^n\left(\frac{\theta}{2n}+\frac{q\pi}{n}\right)\left(\cos\left(\frac{\theta}{2}+q\pi\right)+\sin\left(\frac{\theta}{2}+q\pi\right)i\right)}=\frac{\cos^n\left(\frac{\theta}{2n}+\frac{p\pi}{n}\right)}{\cos^n\left(\frac{\theta}{2n}+\frac{q\pi}{n}\right)}[1;(p-q)\pi]\in IR$$

إذن لكل $p,q\in\{1;2...;n\}$ M_p و M_q مستقيمية مع O، بالتالي النقط M_1 و M_2 و ... و M_n مستقيمية

للتذكير $[r;k\pi]\in IR$ $\quad\forall k\in Z\ \ \forall r\in IR^*$

تمرين 4 : نعتبر العدد : $Z=\cos\alpha+i\sin\alpha$ حيث $\alpha\in\left]-\pi;\pi\right]$

لدينا $|Z|=1$ منه : $Z\bar{Z}=1$ منه : $1+Z+Z^2=Z\bar{Z}+Z+Z^2=Z(\bar{Z}+1+Z)=(1+Z+\bar{Z})Z=(1+2\cos\alpha)Z$

- إذا كان : $\cos\alpha=\frac{-1}{2}$ فإن : $1+Z+Z^2=0$ (لا يمكن كتابته على الشكل المثلثي)
- إذا كان : $\cos\alpha>\frac{-1}{2}$ فإن : $1+Z+Z^2=[1+2\cos\alpha;\alpha]$
- إذا كان : $\cos\alpha<\frac{-1}{2}$ فإن : $1+Z+Z^2=[-(1+2\cos\alpha);\alpha+\pi]$

الشكل المثلثي $r(\cos\alpha+i\sin\alpha)$ يستوجب أن يكون $r>0$ ، فإذا كان سالبا يكون لدينا :

$$r(\cos\alpha+i\sin\alpha)=(-r)\times(-1)\times[1;\alpha]=[-r;0]\times[1;\pi]\times[1;\alpha]=[-r;\alpha+\pi]$$

الشكل الأسي لا يستوجب الشرط السابق