سلسلة 2	نهاية دالة عددية حلول مقترحة	السنة 1 بكالوريا علوم رياضية

تمرين 1 :

$$\lim_{\substack{x\mapsto 1\\ x>1}} \frac{|x-1|+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x>1}} \frac{x-1+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x>1}} \frac{x^2+x-2}{x^2+x-2} = 1$$

$$\lim_{\substack{x\mapsto 1\\ x<1}} \frac{|x-1|+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{1-x+x^2-1}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x^2-x}{x^2+x-2} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x(x-1)}{(x-1)(x+2)} = \lim_{\substack{x\mapsto 1\\ x<1}} \frac{x}{x+2} = \frac{1}{3}$$

لدينا : $x-10 \geq 0 \quad \forall x \in [10+\infty[$ منه :

$$\lim_{x\mapsto +\infty} \frac{|x-10|+x^2}{2x^2+5} = \lim_{x\mapsto +\infty} \frac{x-10+x^2}{2x^2+5} = \lim_{x\mapsto +\infty} \frac{x^2}{2x^2} = \frac{1}{2}$$

يمكننا دائما اعتبار $x \in [a;+\infty[$ عند حساب نهاية في $+\infty$

يمكننا دائما اعتبار $x \in]-\infty;a]$ عند حساب نهاية في $-\infty$

لدينا : $x-1 \geq 0 \quad \forall x \in [1;3]$ منه :

$$\lim_{x\mapsto 2} \frac{|x-1|+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{x-1+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{x^3-8+x-2}{x^2-4}$$

$$\lim_{x\mapsto 2} \frac{|x-1|+x^3-9}{x^2-4} = \lim_{x\mapsto 2} \frac{(x-2)(x^2+2x+4+1)}{(x-2)(x+2)} = \lim_{x\mapsto 2} \frac{x^2+2x+5}{x+2} = \frac{13}{4}$$

يمكننا دائما اعتبار $x \in [a,b]$ حيث $x_0 \in [a,b]$ عند حساب نهاية في x_0

$$\lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}+\sqrt{x}-3}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}-2}{x-1}+\frac{\sqrt{x}-1}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}+\frac{x-1}{(x-1)(\sqrt{x}+1)}\right)$$

$$\lim_{x\mapsto 1} \left(\frac{\sqrt{x+3}+\sqrt{x}-3}{x-1}\right) = \lim_{x\mapsto 1} \left(\frac{1}{\sqrt{x+3}+2}+\frac{1}{\sqrt{x}+1}\right) = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$$

(نضع $t = x-1$)

$$\lim_{x\mapsto 1} \frac{\sin(\text{®}\, x)}{x-1} = \lim_{t\mapsto 0} \frac{\sin(\text{®}\,(t+1))}{t} = \lim_{t\mapsto 0} \frac{\sin(\text{®}\, t+\text{®})}{t} = \lim_{t\mapsto 0} \text{®}\, \frac{\sin(\text{®}\, t)}{\text{®}\, t} = \text{®}$$

$$\lim_{x\mapsto 0} \frac{1-\cos x \cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x+\cos x-\cos x \cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x}{x^2}+\cos x\frac{1-\cos 2x}{x^2}$$

$$\lim_{x\mapsto 0} \frac{1-\cos x \cos 2x}{x^2} = \lim_{x\mapsto 0} \frac{1-\cos x}{x^2}+4\cos x\frac{1-\cos 2x}{(2x)^2} = \frac{1}{2}+4\times 1\times\frac{1}{2} = \frac{5}{2}$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \lim_{x\mapsto 1} \frac{x^3(\sqrt{x+3}-2+2)-2}{x-1} = \lim_{x\mapsto 1} \frac{x^3(\sqrt{x+3}-2)}{x-1}+\frac{2x^3-2}{x-1}$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \lim_{x\mapsto 1} x^3\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}+2\frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\mapsto 1} x^3\frac{1}{\sqrt{x+3}+2}+2(x^2+x+1)$$

$$\lim_{x\mapsto 1} \frac{x^3\sqrt{x+3}-2}{x-1} = \frac{1}{4}+6 = \frac{25}{4}$$

$$\lim_{x\mapsto\frac{®}{4}}\frac{x-\frac{®}{4}\tan^2 x}{x-\frac{®}{4}}=\lim_{x\mapsto\frac{®}{4}}\frac{x-\frac{®}{4}+\frac{®}{4}-\frac{®}{4}\tan^2 x}{x-\frac{®}{4}}=\lim_{x\mapsto\frac{®}{4}}1+\frac{®}{4}\frac{1-\tan^2 x}{x-\frac{®}{4}}=\lim_{x\mapsto\frac{®}{4}}1-\frac{®}{4}(\tan x+1)\frac{\tan x-1}{x-\frac{®}{4}}$$

$$\lim_{x\mapsto\frac{®}{4}}\frac{x-\frac{®}{4}\tan^2 x}{x-\frac{®}{4}}=\lim_{x\mapsto\frac{®}{4}}1-\frac{®}{4}(1+\tan x)\frac{\tan\left(x-\frac{®}{4}\right)(1+\tan x)}{x-\frac{®}{4}}=\lim_{x\mapsto\frac{®}{4}}1-\frac{®}{4}(1+\tan x)^2\frac{\tan\left(x-\frac{®}{4}\right)}{x-\frac{®}{4}}$$

$$\lim_{x\mapsto\frac{®}{4}}\frac{x-\frac{®}{4}\tan^2 x}{x-\frac{®}{4}}=1-\frac{®}{4}\times4\times1=1-®$$

للتذكير : $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$ ما يعني أن : $\tan x-\tan y=\tan(x-y)(1+\tan x\tan y)$

تمرين 2 :

نعلم أن : $2x-1<E(2x)\le 2x$ $\forall x>0$ منه $2-\dfrac{1}{x}<\dfrac{E(2x)}{x}\le 2$ $\forall x>0$ ، بما أن :

$$\lim_{x\mapsto+\infty}2-\frac{1}{x}=\lim_{x\mapsto+\infty}2=2$$

فإن : $\displaystyle\lim_{x\mapsto+\infty}\frac{E(2x)}{x}=2$

لدينا : $\dfrac{E(2x)}{x}=0 \Rightarrow E(2x)=0 \Rightarrow 2x]0;1[\Rightarrow x\in\left]0;\dfrac{1}{2}\right[$ منه : $\displaystyle\lim_{\substack{x\mapsto0\\x>0}}\frac{E(2x)}{x}=\lim_{\substack{x\mapsto0\\x>0}}0=0$

و لدينا : $\dfrac{E(2x)}{x}=\dfrac{-1}{x} \Rightarrow E(2x)=-1 \Rightarrow 2x]-1;0[\Rightarrow x\in\left]-\dfrac{1}{2};0\right[$ منه : $\displaystyle\lim_{\substack{x\mapsto0\\x<0}}\frac{E(2x)}{x}=\lim_{\substack{x\mapsto0\\x<0}}\frac{-1}{x}=+\infty$

تمعن جيدا في الطريقة المتبعة في كل نهاية، فدالة الجزء الصحيح دالة خاصة جدا لذلك نستعمل طرق خاصة بها.

عند حساب النهاية يمين عدد x_0 يمكنك دائما أن تعتبر $x\in]x_0;a[$ (أيضا على اليسار $x\in]a;x_0[$)

لدينا : $E(x)+x=1+x \Rightarrow E(x)=1 \Rightarrow x\in]1;2[$ منه : $\displaystyle\lim_{\substack{x\mapsto1\\x>1}}E(x)+x=\lim_{\substack{x\mapsto1\\x>1}}1+x=2$

و $E(x)+x=x \Rightarrow E(x)=0 \Rightarrow x\in]0;1[$ منه : $\displaystyle\lim_{\substack{x\mapsto1\\x<1}}E(x)+x=\lim_{\substack{x\mapsto1\\x<1}}x=1$

لا يمكننا التعويض ببساطة كما نفعل مع جل الدوال الاعتيادية، فدالة الجزء الصحيح نقول عنها في الرياضيات أنها دالة غير متصلة

لدينا : $E(2x)+\sqrt{x}=2+x \Rightarrow E(2x)=2 \Rightarrow 2x\in]2;3[\Rightarrow x\in\left]1;\dfrac{3}{2}\right[$

منه : $\displaystyle\lim_{\substack{x\mapsto1\\x>1}}E(2x)+\sqrt{x}=\lim_{\substack{x\mapsto1\\x>1}}2+x=3$

و $E(2x)+\sqrt{x}=1+x \Rightarrow E(2x)=1 \Rightarrow 2x\in]1;2[\Rightarrow x\in\left]\dfrac{1}{2};1\right[$ منه : $\displaystyle\lim_{\substack{x\mapsto1\\x<1}}E(2x)+\sqrt{x}=\lim_{\substack{x\mapsto1\\x<1}}1+x=2$

نعلم أن : $\begin{cases}2x-1<E(2x)\le 2x\\ x-1<E(x)\le x\end{cases}$ $\forall x>0$ منه : $\begin{cases}2x-1<E(2x)\le 2x\\ \dfrac{1}{x}<\dfrac{1}{E(x)}\le\dfrac{1}{x-1}\end{cases}$ $\forall x>1$

منه : $\dfrac{2x-1}{x}<\dfrac{E(2x)}{E(x)}\le\dfrac{2x}{x-1}$ $\forall x>1$ وبما أن : $\displaystyle\lim_{x\mapsto+\infty}\frac{2x}{x}=\lim_{x\mapsto+\infty}\frac{2x}{x-1}=\lim_{x\mapsto+\infty}\frac{2x-1}{x}=2$ فإن: $\displaystyle\lim_{x\mapsto+\infty}\frac{E(2x)}{E(x)}=2$

لدينا : $x \in \left]1;\sqrt{2}\right[\Rightarrow \begin{cases} x^2 \in \left]1;2\right[\\ x \in \left]1;2\right[\end{cases} \Rightarrow \begin{cases} E(x^2)=1 \\ E(x)=1 \end{cases} \Rightarrow E(x^2)-(E(x))^2 = 1-1^2 = 0$

منه: $\lim\limits_{\substack{x \mapsto 1 \\ x>1}} E(x^2)-(E(x))^2 = \lim\limits_{\substack{x \mapsto 1 \\ x>1}} 0 = 0$

و لدينا : $x \in \left]0;1\right[\Rightarrow \begin{cases} x^2 \in \left]0;1\right[\\ x \in \left]0;1\right[\end{cases} \Rightarrow \begin{cases} E(x^2)=0 \\ E(x)=0 \end{cases} \Rightarrow E(x^2)-(E(x))^2 = 0-0^2 = 0$

منه: $\lim\limits_{\substack{x \mapsto 1 \\ x<1}} E(x^2)-(E(x))^2 = \lim\limits_{\substack{x \mapsto 1 \\ x<1}} 0 = 0$ ، خلاصة: $\lim\limits_{x \mapsto 1} E(x^2)-(E(x))^2 = 0$

تمرين 3 : احسب النهايات التالية :

$$\lim_{\substack{x \mapsto 0 \\ x>0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\left|\frac{\cos 3x - 1}{x^2} + \frac{1-\cos x}{x^2}\right|}$$

$$\lim_{\substack{x \mapsto 0 \\ x>0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x>0}} \sqrt{\left|-9\frac{1-\cos 3x}{(3x)^2} + \frac{1-\cos x}{x^2}\right|} = \sqrt{\left|\frac{-9}{2} + \frac{1}{2}\right|} = 2$$

$$\lim_{\substack{x \mapsto 0 \\ x<0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \mapsto 0 \\ x<0}} -\sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = -2$$

$\forall x > 0 \ \ x = x^2$ و $\forall x < 0 \ \ x = -x^2$

$$\lim_{x \mapsto +\infty} \sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} = \lim_{x \mapsto +\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}} = \lim_{x \mapsto +\infty} \frac{\sqrt{x\left(1+\frac{\sqrt{x}}{x}\right)}}{\sqrt{x\left(1+\frac{\sqrt{x+\sqrt{x}}}{x}\right)} + \sqrt{x}}$$

$$\lim_{x \mapsto +\infty} \sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} = \lim_{x \mapsto +\infty} \frac{\sqrt{x}\sqrt{1+\frac{\sqrt{x}}{x}}}{\sqrt{x}\left(\sqrt{1+\sqrt{\frac{x+\sqrt{x}}{x^2}}} + 1\right)} = \lim_{x \mapsto +\infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x\sqrt{x}}}} + 1} = \frac{1}{2}$$

يمكن وضع $t = x^2$ للتبسيط

$$\lim_{x \mapsto -\infty} x\sqrt{x^2+1} + x^2 = \lim_{t \mapsto +\infty} -t\sqrt{t^2+1} + t^2 = \lim_{t \mapsto +\infty} t^2 - t\sqrt{t^2+1} = \lim_{t \mapsto +\infty} \frac{t^4 - t^4 - t^2}{t^2 + t\sqrt{t^2+1}}$$

(نضع $t = -x$)

$$\lim_{x \mapsto -\infty} x\sqrt{x^2+1} + x^2 = \lim_{t \mapsto +\infty} \frac{-t^2}{t^2 + t^2\sqrt{1+\frac{1}{t^2}}} = \lim_{t \mapsto +\infty} \frac{-1}{1+\sqrt{1+\frac{1}{t^2}}} = \frac{-1}{2}$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\sqrt{1-\cos(x)}-\sqrt{1-\sin(x)}}{1-\tan(x)}\right)=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sin(x)-\cos(x)}{(1-\tan(x))\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(x)-\frac{1}{\sqrt{2}}\cos(x)\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\sin(x)-\sin\left(\frac{\pi}{4}\right)\cos(x)\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{\left(\tan\left(\frac{\pi}{4}-x\right)(1+\tan(x))\right)\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}$$

$$=\lim_{x\mapsto\frac{\pi}{4}}\frac{-\sqrt{2}}{(1+\tan(x))\left(\sqrt{1-\cos(x)}+\sqrt{1-\sin(x)}\right)}\left(\frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}\right)\left(\frac{x-\frac{\pi}{4}}{\tan\left(x-\frac{\pi}{4}\right)}\right)$$

$$=\frac{-\sqrt{2}}{2\times\left(\sqrt{1-\frac{\sqrt{2}}{2}}+\sqrt{1-\frac{\sqrt{2}}{2}}\right)}\times1\times1=\frac{-\sqrt{2}}{4\sqrt{\frac{2-\sqrt{2}}{2}}}=\frac{-2}{4\sqrt{2-\sqrt{2}}}$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\sqrt{1-\cos(x)}-\sqrt{1-\sin(x)}}{1-\tan(x)}\right)=\frac{-1}{2}\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}}=\frac{-\sqrt{4+2\sqrt{2}}}{4}$$

تمرين 4 :

$$\lim_{x\mapsto a}\frac{x\sqrt{x}-a\sqrt{a}}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}\frac{\left(\sqrt{x}\right)^3-\left(\sqrt{a}\right)^3}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}\frac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}^2+\sqrt{x}\sqrt{a}+\sqrt{a}^2\right)}{\sqrt{x}-\sqrt{a}}=\lim_{x\mapsto a}x+\sqrt{a\,x}+a=3a$$

$$\lim_{x\mapsto1}\frac{x^{2015}-1-2015(x-1)}{(x-1)^2}=\lim_{x\mapsto1}\frac{(x-1)\left(x^{2014}+x^{2013}+\ldots+x+1\right)-2015(x-1)}{(x-1)^2}$$

$$=\lim_{x\mapsto1}\frac{x^{2014}+x^{2013}+\ldots+x+1-2015}{x-1}$$

$$=\lim_{x\mapsto1}\frac{x^{2014}+x^{2013}+\ldots+x-2014}{x-1}=\lim_{x\mapsto1}\frac{x^{2014}-1}{x-1}+\frac{x^{2013}-1}{x-1}+\ldots+\frac{x-1}{x-1}$$

$$=\lim_{x\mapsto1}\left(x^{2013}+x^{2012}+\ldots+x+1\right)+\left(x^{2012}+x^{2011}+\ldots+x+1\right)+\ldots+1$$

$$\lim_{x\mapsto1}\frac{x^{2015}-1-2015(x-1)}{(x-1)^2}=2014+2013+\ldots+1=\frac{2014\times2015}{2}=2029105$$