| السنة 1 بكالوريا علوم رياضية | نهاية دالة عددية<br>حلول مقترحة | رياضيات النجاح |
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**تمرين 1** :

$$\lim_{x\mapsto+\infty}\left(\sqrt{9x^2+x+1}-2x\right)=\lim_{x\mapsto+\infty}\left(\sqrt{x^2\left(9+\frac{1}{x}+\frac{1}{x^2}\right)}-2x\right)=\lim_{x\mapsto+\infty}\left(x\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2x\right)$$

$$\lim_{x\mapsto+\infty}\left(\sqrt{9x^2+x+1}-2x\right)=\lim_{x\mapsto+\infty}x\left(\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-2=3-2=1$ و $\lim_{x\mapsto+\infty}x=+\infty$ )

$$\lim_{x\mapsto-\infty}\left(\sqrt{9x^2+x+1}+3x\right)=\lim_{x\mapsto-\infty}\frac{9x^2+x+1-9x^2}{\sqrt{9x^2+x+1}-3x}=\lim_{x\mapsto-\infty}\frac{x+1}{\sqrt{x^2\left(9+\frac{1}{x}+\frac{1}{x^2}\right)}-3x}$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{9x^2+x+1}+3x\right)=\lim_{x\mapsto-\infty}\frac{x\left(1+\frac{1}{x}\right)}{-x\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-3x}=\lim_{x\mapsto-\infty}\frac{1+\frac{1}{x}}{-\sqrt{9+\frac{1}{x}+\frac{1}{x^2}}-3}=\frac{1}{-3-3}=\frac{-1}{6}$$

- انتبه أنه إذا كان : $x<0$ فإن : $\sqrt{x^2}=|-x|=-x$
- الطريقة السابقة لن تجدي في هذه النهاية لأننا سنحصل على شكل غير محدد
- المثالان متشابهان لكن الطريقة تختلف و السبب المعاملات الموجودة في كل نهاية، لذلك نأمل أن تدرك متى نستعمل المرافق و متى نعمل بأكبر أس.

$$\lim_{x\mapsto+\infty}\left(x^2-\sqrt{x+2}\right)=\lim_{x\mapsto+\infty}\left(x^2-\sqrt{x^2\left(\frac{1}{x}+\frac{2}{x}\right)}\right)=\lim_{x\mapsto+\infty}\left(x^2-x\sqrt{\frac{1}{x}+\frac{2}{x}}\right)=\lim_{x\mapsto+\infty}x\left(x-\sqrt{\frac{1}{x}+\frac{2}{x}}\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}x-\sqrt{\frac{1}{x}+\frac{2}{x}}=+\infty$ و $\lim_{x\mapsto+\infty}x=+\infty$ )

$$\lim_{x\mapsto+\infty}\sqrt{1+2x^3}-\sqrt{x^3+x+1}=\lim_{x\mapsto+\infty}\sqrt{x^3}\sqrt{\frac{1}{x^3}+2}-\sqrt{x^3}\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}$$

$$\lim_{x\mapsto+\infty}\sqrt{1+2x^3}-\sqrt{x^3+x+1}=\lim_{x\mapsto+\infty}\sqrt{x^3}\left(\sqrt{\frac{1}{x^3}+2}-\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}\right)=+\infty$$

(لأن : $\lim_{x\mapsto+\infty}\sqrt{\frac{1}{x^3}+2}-\sqrt{1+\frac{1}{x^2}+\frac{1}{x^3}}=\sqrt{2}-1>0$ و $\lim_{x\mapsto+\infty}\sqrt{x^3}=+\infty$ )

- إذا كان $m>0$ فإن : $\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=+\infty$ ( $\lim_{x\mapsto-\infty}-mx=+\infty$ )
- إذا كان $m=0$ فإن : $\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}\right)+\infty$
- إذا كان $-2<m<0$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\left(\sqrt{x^2\left(4+\frac{3}{x}+\frac{7}{x^2}\right)}-mx\right)=\lim_{x\mapsto-\infty}\left(-x\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-mx\right)$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}-x\left(\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}+m\right)=+\infty \qquad (+\infty\times(2+m)>0)$$

- إذا كان $m=-2$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\sqrt{4x^2+3x+7}+2x=\lim_{x\mapsto-\infty}\frac{3x+7}{\sqrt{4x^2+3x+7}-2x}$$

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}\frac{x\left(3+\frac{7}{x}\right)}{-x\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-2x}=\lim_{x\mapsto-\infty}\frac{3+\frac{7}{x}}{-\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}-2}=\frac{-3}{4}$$

إذا كان $m<-2$ فإن :

$$\lim_{x\mapsto-\infty}\left(\sqrt{4x^2+3x+7}-mx\right)=\lim_{x\mapsto-\infty}-x\left(\sqrt{4+\frac{3}{x}+\frac{7}{x^2}}+m\right)=-\infty\qquad\left(+\infty\times(2+m)<0\right)$$

سؤال يوضح بالتفاصيل متى نستعمل طريقة النهاية الأولى و متى نستعمل المرافق

$$\lim_{x\mapsto+\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto+\infty}\frac{\left(x^2-\left(x^2+x+1\right)\right)\left(x^2+\sqrt{x^4-1}\right)}{\left(x^4-\left(x^4-1\right)\right)\left(x+\sqrt{x^2+x+1}\right)}=\lim_{x\mapsto+\infty}\frac{-(x+1)\left(x^2+x^2\sqrt{1-\frac{1}{x^4}}\right)}{\left(x+x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\right)}$$

$$\lim_{x\mapsto+\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)==\lim_{x\mapsto+\infty}\frac{-x(x+1)\left(1+\sqrt{1-\frac{1}{x^4}}\right)}{1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}}=-\infty$$

$$\lim_{x\mapsto-\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto-\infty}\frac{\left(x-\sqrt{x^2+x+1}\right)\left(x^2+\sqrt{x^4-1}\right)}{\left(x^4-\left(x^4-1\right)\right)}=\lim_{x\mapsto-\infty}\left(x-\sqrt{x^2+x+1}\right)\left(x^2+x^2\sqrt{1-\frac{1}{x^4}}\right)$$

$$\lim_{x\mapsto-\infty}\left(\frac{x-\sqrt{x^2+x+1}}{x^2-\sqrt{x^4-1}}\right)=\lim_{x\mapsto-\infty}x^2\left(x-\sqrt{x^2+x+1}\right)\left(1+\sqrt{1-\frac{1}{x^4}}\right)=-\infty$$

(لأن : $(-\infty+-\infty)$ $\lim_{x\mapsto-\infty}x-\sqrt{x^2+x+1}=-\infty$ و $\lim_{x\mapsto-\infty}x^2=+\infty$ و $\lim_{x\mapsto-\infty}1+\sqrt{1-\frac{1}{x^4}}=1$ )

$$\lim_{x\mapsto-\infty}\left(\frac{x+1-\sqrt{1-x}}{x^2-\sqrt{x^2+2}}\right)=\lim_{x\mapsto-\infty}\frac{x+1-\sqrt{x^2\left(\frac{1}{x^2}-\frac{1}{x}\right)}}{x^2-\sqrt{x^2\left(1+\frac{2}{x^2}\right)}}=\lim_{x\mapsto-\infty}\frac{x+1+x\sqrt{\frac{1}{x^2}-\frac{1}{x}}}{x^2+x\sqrt{1+\frac{2}{x^2}}}$$

$$=\lim_{x\mapsto-\infty}\frac{x\left(1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}\right)}{x\left(x+\sqrt{1+\frac{2}{x^2}}\right)}=\lim_{x\mapsto-\infty}\frac{1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}}{x+\sqrt{1+\frac{2}{x^2}}}=0$$

(لأن : $\lim_{x\mapsto+\infty}x+\sqrt{1+\frac{2}{x^2}}=-\infty$ و $\lim_{x\mapsto+\infty}1+\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{x}}=1$ )

**تمرين 2** :

$$\lim_{x\mapsto-1}\left(\frac{\sqrt{1-3x}-2}{x+1}\right)=\lim_{x\mapsto-1}\frac{1-3x-4}{(x+1)\left(\sqrt{1-3x}+2\right)}=\lim_{x\mapsto-1}\frac{-3}{\sqrt{1-3x}+2}=\frac{-3}{4}$$

$$\lim_{x\mapsto2^+}\frac{\sqrt{x^2-4}}{x-2}=\lim_{x\mapsto2^+}\sqrt{\frac{(x-2)(x+2)}{(x-2)^2}}=\lim_{x\mapsto2^+}\sqrt{\frac{x+2}{x-2}}=+\infty$$

$$\lim_{x\mapsto -2^-}\frac{\sqrt{x^2-4}}{x+2}=\lim_{x\mapsto 2^-}-\frac{\sqrt{x^2-4}}{-(x+2)}=\lim_{x\mapsto 2^+}-\sqrt{\frac{(x-2)(x+2)}{(x+2)^2}}=\lim_{x\mapsto 2^+}-\sqrt{\frac{x-2}{x+2}}=-\infty$$

$$\lim_{x\mapsto 1}\left(\frac{x^2+x-2}{x^3+4x^2-8x+3}\right)=\lim_{x\mapsto 1}\frac{(x-1)(x+2)}{(x-1)(x^2+5x-3)}=1$$

استعملنا القسمة الإقليدية على $(x-1)$ للتعميل في البسط و المقام

$$\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}+\sqrt{3-x}-3}{x+1}\right)=\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}-1}{x+1}+\frac{\sqrt{3-x}-2}{x+1}\right)$$

$$=\lim_{x\mapsto -1}\left(\frac{x+1}{(x+1)(\sqrt{2+x}+1)}+\frac{-1-x}{(x+1)(\sqrt{3-x}+2)}\right)$$

$$=\lim_{x\mapsto -1}\left(\frac{1}{\sqrt{2+x}+1}+\frac{-1}{\sqrt{3-x}+2}\right)$$

$$\lim_{x\mapsto -1}\left(\frac{\sqrt{2+x}+\sqrt{3-x}-3}{x+1}\right)=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$\lim_{x\mapsto 0^+}\left(\frac{x+\sqrt{x}}{\sqrt{x^2+x}-x}\right)=\lim_{x\mapsto 0^+}\left(\frac{\sqrt{x}^2+\sqrt{x}}{\sqrt{x(x+1)}-x}\right)=\lim_{x\mapsto 0^+}\frac{\sqrt{x}(\sqrt{x}+1)}{\sqrt{x}\sqrt{x+1}-\sqrt{x}^2}=\lim_{x\mapsto 0^+}\frac{\sqrt{x}(\sqrt{x}+1)}{\sqrt{x}(\sqrt{x+1}-\sqrt{x})}$$

$$\lim_{x\mapsto 0^+}\left(\frac{x+\sqrt{x}}{\sqrt{x^2+x}-x}\right)=\lim_{x\mapsto 0^+}\frac{\sqrt{x}+1}{\sqrt{x+1}-\sqrt{x}}=1$$

**تمرين 3 :**

$$\lim_{x\mapsto 0}\left(\frac{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}{x^2}\right)=\lim_{x\mapsto 0}\left(-\sqrt{3}\left(\frac{1-\cos(x)}{x^2}\right)-\left(\frac{\sin(x)}{x}\right)^2\right)=\frac{-\sqrt{3}}{2}-1$$

$$\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}\right)=\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}(\cos x-1)-\sin^2 x}\right)=\lim_{x\mapsto 0}\left(\frac{-1}{\sqrt{3}(1-\cos(x))+\sin^2 x}\right)$$

$$\lim_{x\mapsto 0}\left(\frac{1}{\sqrt{3}\cos x-\sin^2 x-\sqrt{3}}\right)=\lim_{x\mapsto 0}\left(\frac{-\frac{1}{x^2}}{\sqrt{3}\left(\frac{1-\cos(x)}{x^2}\right)+\left(\frac{\sin x}{x}\right)^2}\right)=-\infty$$

للتذكير : $\lim_{x\mapsto 0}\frac{\sin x}{x}=1$ و $\lim_{x\mapsto 0}\frac{1-\cos x}{x}=1$

لدينا :

$$\lim_{x\mapsto \frac{®}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\lim_{x\mapsto \frac{®}{4}}\left(\frac{\tan x-\tan\left(\frac{®}{4}\right)}{2\cos x-\sqrt{2}}\right)=\lim_{x\mapsto \frac{®}{4}}\left(\frac{\left(1+\tan(x)\tan\left(\frac{®}{4}\right)\right)\tan\left(x-\frac{®}{4}\right)}{2\cos x-\sqrt{2}}\right)$$

الآن نضع $t = x-\frac{®}{4}$ فنجد :

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\lim_{t\mapsto 0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{2\cos\left(t+\frac{\pi}{4}\right)-\sqrt{2}}\right)=\lim_{t\mapsto 0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{2\left(\cos(t)\frac{\sqrt{2}}{2}-\sin(t)\frac{\sqrt{2}}{2}\right)-\sqrt{2}}\right)$$

$$=\lim_{t\mapsto 0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\tan(t)}{-\sqrt{2}(1-\cos(t))-\sqrt{2}\sin(t)}\right)=\lim_{t\mapsto 0}\left(\frac{\left(1+\tan\left(t+\frac{\pi}{4}\right)\right)\frac{\tan(t)}{t}}{-\sqrt{2}\,t\frac{(1-\cos(t))}{t^2}-\sqrt{2}\frac{\sin(t)}{t}}\right)$$

$$\lim_{x\mapsto\frac{\pi}{4}}\left(\frac{\tan x-1}{2\cos x-\sqrt{2}}\right)=\frac{(1+1)\times 1}{-\sqrt{2}\times 0\times\frac{1}{2}-\sqrt{2}\times 1}=-\sqrt{2}$$

لاحظ أن استعمال الخاصية $\tan(a-b)=\frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}$

بالشكل $\tan(a)-\tan(b)=(1+\tan(a)\tan(b))\tan(a-b)$ أفضل من استعمال تغيير المتغير من البداية.

$$\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{\cos x-\sqrt{3}\sin x}{6x-\pi}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{2\left(\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x\right)}{6\left(x-\frac{\pi}{6}\right)}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{1}{3}\frac{\left(\sin\left(\frac{\pi}{6}\right)\cos x-\cos\left(\frac{\pi}{6}\right)\sin x\right)}{\left(x-\frac{\pi}{6}\right)}\right)$$

$$\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{\cos x-\sqrt{3}\sin x}{6x-\pi}\right)=\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{1}{3}\frac{\sin\left(\frac{\pi}{6}-x\right)}{\left(x-\frac{\pi}{6}\right)}\right)==\lim_{x\mapsto\frac{\pi}{6}}\left(\frac{-1}{3}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}\right)=\lim_{t\mapsto 0}\left(\frac{-1}{3}\frac{\sin(t)}{t}\right)=\frac{-1}{3}$$

(قمنا بتغيير المتغير $x$ وذلك بوضع $t=x-\frac{\pi}{6}$)، كما يمكن إجراء تغيير المتغير منذ البداية.

**تمرين 4** : $f(x)=x^2\sin\left(\frac{1}{x^2}\right)$

لدينا : $-1\leq\sin\left(\frac{1}{x^2}\right)\leq 1$ $\forall x\in IR^*$ منه : $-x^2\leq f(x)\leq x^2$ $\forall x\in IR^*$

بما أن : $\lim_{x\mapsto 0}x^2=0$ و $\lim_{x\mapsto 0}-x^2=0$ فإن : $\lim_{x\mapsto 0}f(x)=0$

نضع : $t=\frac{1}{x^2}$ إذن: $\lim_{x\mapsto+\infty}f(x)=\lim_{x\mapsto+\infty}x^2\sin\left(\frac{1}{x^2}\right)=\lim_{t\mapsto 0}\frac{1}{t}\sin(t)=1$