

الأستاذ:
نجيب
عثماني

تمارين محلولة: الحساب المثلثي
المستوى : الأولى باك علوم تجريبية

أكاديمية
الجهة
الشرقية

تمرين 1: أحسب $\sin \frac{\pi}{12}$ و $\cos \frac{\pi}{12}$
أجوبة: $\cos \frac{\pi}{12} = \cos \left(\frac{4\pi - 3\pi}{12} \right) = \cos \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$

$$\cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

يمكننا استعمال نتائج الجدول التالي:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

$$\cos \frac{\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3}$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

تمرين 2: أحسب $\tan \frac{\pi}{12}$

الجواب: $\tan \frac{\pi}{12} = \tan \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \times \tan \frac{\pi}{4}} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$

$$\tan \frac{\pi}{12} = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3})^2 - 1^2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

تمرين 3:

1. أحسب $\tan \frac{5\pi}{12}$ و $\sin \frac{5\pi}{12}$ و $\cos \frac{5\pi}{12}$

2. أحسب $\tan \frac{7\pi}{12}$ و $\sin \frac{7\pi}{12}$ و $\cos \frac{7\pi}{12}$

3. بين أن : $\cos x = \cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right)$

أجوبة: (1) $\cos \frac{5\pi}{12} = \cos \left(\frac{2\pi + 3\pi}{12} \right) = \cos \left(\frac{2\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{6} + \frac{\pi}{4} \right)$

$$\cos \left(\frac{\pi}{6} + \frac{\pi}{4} \right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$$

$$\cos \frac{5\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{6} + \frac{\pi}{4} \right) = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \frac{\pi}{6}$$

$$\sin \frac{5\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

(1) قواعد مهمة : يجب حفظها

$$\textcircled{1} \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\textcircled{2} \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\textcircled{3} \sin(a+b) = \sin a \cos b + \sin b \cos a$$

$$\textcircled{4} \sin(a-b) = \sin a \cos b - \sin b \cos a$$

(2)

$$\textcircled{5} \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \times \tan b}$$

$$\textcircled{6} \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \times \tan b}$$

$$\cos(2a) = 1 - 2\sin^2 a \quad \text{و} \quad \cos(2a) = \cos^2 a - \sin^2 a \quad (3)$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} \quad \text{اذن} \quad \cos(2a) = 2\cos^2 a - 1$$

$$\cos^2 a + \sin^2 a = 1 \quad \text{و} \quad \sin^2 a = \frac{1 - \cos 2a}{2}$$

$$1 + \tan^2 a = \frac{1}{\cos^2 a} \quad \text{و} \quad \sin(2a) = 2\sin a \times \cos a$$

$$\tan(x) = \frac{2 \tan \left(\frac{x}{2} \right)}{1 - \tan^2 \left(\frac{x}{2} \right)} \quad \tan(2a) = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\sin x = 2 \sin \frac{x}{2} \times \cos \frac{x}{2} \quad \text{و} \quad \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \quad (4)$$

$$\sin x = \frac{2 \tan^2 \left(\frac{x}{2} \right)}{1 + \tan^2 \left(\frac{x}{2} \right)} \quad \text{و} \quad \cos x = \frac{1 - \tan^2 \left(\frac{x}{2} \right)}{1 + \tan^2 \left(\frac{x}{2} \right)}$$

(5) بوضع : $t = \tan \left(\frac{x}{2} \right)$

نجد : $\tan x = \frac{2t}{1-t^2}$ و $\cos x = \frac{1-t^2}{1+t^2}$ و $\sin x = \frac{2t}{1+t^2}$

(6) قواعد لكيفية تحويل جداء إلى مجموع:

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\sin a \sin b = -\frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \cos b = \frac{1}{2} [\sin(a+b) + \sin(a-b)]$$

$$\cos a \sin b = -\frac{1}{2} [\sin(a+b) - \sin(a-b)]$$

(7) قواعد لكيفية تحويل مجموع إلى جداء:

$$\cos p + \cos q = 2 \cos \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\cos p - \cos q = -2 \sin \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

$$\sin p + \sin q = 2 \sin \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\sin p - \sin q = 2 \cos \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

نعلم أن: $\sin^2 a = 1 - \cos^2 a$ يعني $\cos^2 a + \sin^2 a = 1$ يعني $\sin^2 a = 1 - \left(\frac{1}{2}\right)^2$

يعني $\sin^2 a = \frac{3}{4}$ يعني $\sin a = \frac{\sqrt{3}}{2}$ أو $\sin a = -\frac{\sqrt{3}}{2}$ ونعلم أن: $0 < a < \frac{\pi}{2}$

اذن: $\sin a = \frac{\sqrt{3}}{2}$

نعلم أن: $\sin(a+b) = \sin a \cos b + \sin b \cos a$

اذن: $\sin(a+b) = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$

تمرين 6: علما أن: $\sin x = \frac{1}{3}$ و $x \in \left]0; \frac{\pi}{2}\right[$

أحسب $\sin(2x)$ و $\cos(2x)$

أجوبة: نعلم أن: $\cos(2x) = 1 - 2\sin^2 x$

اذن: $\cos(2x) = 1 - 2\left(\frac{1}{3}\right)^2 = 1 - \frac{2}{9} = \frac{7}{9}$

و نعلم أن: $\sin(2x) = 2\sin x \cos x$ ومنه يجب حساب $\cos x$:

لدينا: $\cos^2 x + \sin^2 x = 1$ يعني $\cos^2 x = 1 - \sin^2 x$ يعني $\cos^2 x = 1 - \left(\frac{1}{3}\right)^2$

يعني $\cos^2 x = \frac{8}{9}$ يعني $\cos x = \frac{\sqrt{8}}{3}$ أو $\cos x = -\frac{\sqrt{8}}{3}$ ونعلم أن: $x \in \left]0; \frac{\pi}{2}\right[$

اذن: $\cos x = \frac{\sqrt{8}}{3}$ ومنه: $\sin(2x) = 2 \times \frac{1}{3} \times \frac{\sqrt{8}}{3} = \frac{2\sqrt{8}}{9}$

تمرين 7: أحسب $\sin \frac{\pi}{8}$ و $\cos \frac{\pi}{8}$ (لاحظ أن $\frac{\pi}{4} = 2 \times \frac{\pi}{8}$)

أجوبة: حساب $\cos \frac{\pi}{8}$:

نستعمل العلاقة: $\cos^2 a = \frac{1 + \cos 2a}{2}$ ونضع مثلا: $a = \frac{\pi}{8}$

ونجد: $\cos^2 \frac{\pi}{8} = \frac{2 + \sqrt{2}}{4}$ يعني $\cos^2 \frac{\pi}{8} = \frac{1 + \cos \frac{\pi}{4}}{2}$ يعني $\cos^2 \frac{\pi}{8} = \frac{1 + \cos 2 \frac{\pi}{8}}{2}$

يعني $\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}}$ أو $\cos \frac{\pi}{8} = -\sqrt{\frac{2 + \sqrt{2}}{4}}$

ولكننا نعلم أن: $0 \leq \frac{\pi}{8} \leq \frac{\pi}{2}$ اذن: $\cos \frac{\pi}{8} \geq 0$ ومنه: $\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}}$

حساب $\sin \frac{\pi}{8}$: يمكننا استعمال احدى القواعد التالية: $\sin^2 a = \frac{1 - \cos 2a}{2}$ أو

$\sin(2a) = 2\sin a \cos a$ أو $\cos^2 a + \sin^2 a = 1$

لدينا: $\sin^2 a = \frac{1 - \cos 2a}{2}$ ونضع مثلا: $a = \frac{\pi}{8}$

ونجد: $\sin^2 \frac{\pi}{8} = \frac{2 - \sqrt{2}}{4}$ يعني $\sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2}$ يعني $\sin^2 \frac{\pi}{8} = \frac{1 - \cos 2 \frac{\pi}{8}}{2}$

يعني $\sin \frac{\pi}{8} = \sqrt{\frac{2 - \sqrt{2}}{4}}$ أو $\sin \frac{\pi}{8} = -\sqrt{\frac{2 - \sqrt{2}}{4}}$

ولكننا نعلم أن: $0 \leq \frac{\pi}{8} \leq \frac{\pi}{2}$ اذن: $\sin \frac{\pi}{8} \geq 0$ ومنه: $\sin \frac{\pi}{8} = \sqrt{\frac{2 - \sqrt{2}}{4}}$

تمرين 8: بين أن: $\forall x \in \left]0; \frac{\pi}{2}\right[\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

الجواب: $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{\sin 3x \cos x - \sin x \cos 3x}{\sin x \cos x} = \frac{\sin(3x - x)}{\sin x \cos x}$

$$\tan \frac{5\pi}{12} = \frac{\sin \frac{5\pi}{12}}{\cos \frac{5\pi}{12}} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{(\sqrt{6} + \sqrt{2})^2}{6 - 2} = \frac{(\sqrt{6} + \sqrt{2})^2}{4}$$

$$\tan \frac{5\pi}{12} = \frac{(\sqrt{6} + \sqrt{2})^2}{4} = \frac{8 + 2\sqrt{12}}{4} = \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}$$

$$\cos \frac{7\pi}{12} = \cos \left(\frac{4\pi + 3\pi}{12} \right) = \cos \left(\frac{4\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right)$$

$$\cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

$$\cos \frac{7\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{3} + \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \frac{\pi}{3}$$

$$\sin \frac{7\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\tan \frac{7\pi}{12} = \frac{\sin \frac{7\pi}{12}}{\cos \frac{7\pi}{12}} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{2} - \sqrt{6}}{4}} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}} = \frac{(\sqrt{6} + \sqrt{2})^2}{2 - 6} = \frac{(\sqrt{6} + \sqrt{2})^2}{-4}$$

$$\tan \frac{7\pi}{12} = \frac{8 + 2\sqrt{12}}{-4} = \frac{8 + 4\sqrt{3}}{-4} = -2 - \sqrt{3}$$

$$\textcircled{?} \cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right) = \cos x$$

$$\cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right) = \cos \frac{\pi}{3} \cos x - \sin \frac{\pi}{3} \sin x + \cos \frac{\pi}{3} \cos x + \sin \frac{\pi}{3} \sin x$$

$$= \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x = 2 \times \frac{1}{2} \cos x = \cos x$$

تمرين 4: بين أن: $\sin \left(x + \frac{2\pi}{3} \right) + \sin \left(x - \frac{2\pi}{3} \right) + \sin x = 0$

الجواب: لدينا

$$\sin \left(x + \frac{2\pi}{3} \right) = \sin x \cos \frac{2\pi}{3} + \sin \frac{2\pi}{3} \cos x = \sin x \cos \left(\pi - \frac{\pi}{3} \right) + \sin \left(\pi - \frac{\pi}{3} \right) \cos x$$

$$\sin \left(x + \frac{2\pi}{3} \right) = -\sin x \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cos x$$

$$\sin \left(x - \frac{2\pi}{3} \right) = \sin x \cos \frac{2\pi}{3} - \sin \frac{2\pi}{3} \cos x = \sin x \cos \left(\pi - \frac{\pi}{3} \right) - \sin \left(\pi - \frac{\pi}{3} \right) \cos x$$

$$\sin \left(x - \frac{2\pi}{3} \right) = -\sin x \cos \frac{\pi}{3} - \sin \frac{\pi}{3} \cos x$$

$$\sin \left(x + \frac{2\pi}{3} \right) + \sin \left(x - \frac{2\pi}{3} \right) + \sin x = -2\sin x \cos \frac{\pi}{3} + \sin x = -\sin x + \sin x = 0$$

تمرين 5: علما أن: $0 < a < \frac{\pi}{2}$ و $0 < b < \frac{\pi}{2}$ و $\cos a = \sin b = \frac{1}{2}$

1. أحسب $\sin a$ و $\cos b$

2. أحسب $\sin(a+b)$

أجوبة: 1) حساب $\cos b$

نعلم أن: $\cos^2 b + \sin^2 b = 1$ يعني $\cos^2 b = 1 - \sin^2 b$ يعني $\cos^2 b = 1 - \left(\frac{1}{2}\right)^2$

يعني $\cos^2 b = \frac{3}{4}$ يعني $\cos b = \frac{\sqrt{3}}{2}$ أو $\cos b = -\frac{\sqrt{3}}{2}$ ونعلم أن: $0 < b < \frac{\pi}{2}$

اذن: $\cos b = \frac{\sqrt{3}}{2}$

حساب $\sin a$

$$= \frac{1}{4}(2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x + 3\cos x) = \frac{1}{4}(4\cos^3 x) = \cos^3 x$$

طريقة 2: نستعمل صيغة تحويل جداء الى مجموع

$$\cos^3 x = \cos^2 x \times \cos x = \frac{1+\cos 2x}{2} \times \cos x = \frac{1}{2}(\cos x + \cos 2x \times \cos x)$$

$$\cos^3 x = \frac{1}{2}\left(\cos x + \frac{1}{2}(\cos 3x + \cos x)\right) = \frac{1}{2}\cos x + \frac{1}{4}\cos 3x + \frac{1}{4}\cos x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x) \quad \text{ومنه:}$$

تمرين 12: علما أن $Q(x) = 1 + \cos x + \cos 2x$ و $P(x) = \sin 2x - \sin x$

بين أن $P(x) = \sin x(2\cos x - 1)$ و $Q(x) = \cos x(2\cos x + 1)$

أجوبة:

$$Q(x) = 1 + \cos x + \cos 2x = 1 + \cos x + 2\cos^2 x - 1 = \cos x + 2\cos^2 x = \cos x(1 + 2\cos x)$$

$$P(x) = \sin 2x - \sin x = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)$$

تمرين 13:

أكتب على شكل مجموع :

$$\cos 4x \times \cos 6x \quad (3) \quad \sin x \times \sin 3x \quad (2) \quad \cos 2x \times \sin 4x \quad (1)$$

أجوبة (1):

$$\cos 2x \times \sin 4x = \frac{1}{2}(\sin(2x+4x) - \sin(2x-4x)) = \frac{1}{2}(\sin 6x - \sin(-2x))$$

$$= \frac{1}{2}(\sin 6x + \sin 2x) = \frac{1}{2}\sin 6x + \frac{1}{2}\sin 2x$$

$$\sin x \times \sin 3x = \frac{1}{2}(\cos(x+3x) - \cos(x-3x)) = \frac{1}{2}(\cos 4x - \cos(-2x)) \quad (2)$$

$$\sin x \times \sin 3x = \frac{1}{2}(\cos 4x - \cos(2x)) = \frac{1}{2}\cos 4x - \frac{1}{2}\cos(2x)$$

$$\cos 4x \times \cos 6x = \frac{1}{2}(\cos(4x+6x) + \cos(4x-6x)) = \frac{1}{2}(\cos 10x + \cos(-2x)) \quad (3)$$

$$\cos 4x \times \cos 6x = \frac{1}{2}\cos 10x + \frac{1}{2}\cos(2x)$$

تمرين 14: أكتب على شكل جداء: $\sin 2x + \sin 4x$

$$\sin 2x + \sin 4x = 2\sin\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) \quad \text{الجواب:}$$

$$\sin 2x + \sin 4x = 2\sin 3x \cos(-2x) = 2\sin 3x \cos 2x$$

تمرين 15:

$$1. \text{ بين } \sin\frac{3\pi}{11} + \sin\frac{7\pi}{11} = 2\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{2\pi}{11}\right)$$

$$2. \text{ بين } \sin\frac{3\pi}{11} - \sin\frac{7\pi}{11} = -2\cos\left(\frac{5\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)$$

$$3. \text{ استنتج أن: } \frac{\sin\frac{3\pi}{11} + \sin\frac{7\pi}{11}}{\sin\frac{3\pi}{11} - \sin\frac{7\pi}{11}} = \frac{\tan\left(\frac{5\pi}{11}\right)}{\tan\left(\frac{2\pi}{11}\right)}$$

$$\sin\frac{3\pi}{11} + \sin\frac{7\pi}{11} = 2\sin\left(\frac{3\pi + 7\pi}{2}\right)\cos\left(\frac{3\pi - 7\pi}{2}\right) \quad (1) \quad \text{أجوبة:}$$

$$\sin\frac{3\pi}{11} + \sin\frac{7\pi}{11} = 2\sin\left(\frac{5\pi}{11}\right)\cos\left(-\frac{2\pi}{11}\right) = 2\sin\frac{5\pi}{11}\cos\frac{2\pi}{11}$$

$$\sin\frac{3\pi}{11} - \sin\frac{7\pi}{11} = 2\cos\left(\frac{3\pi + 7\pi}{2}\right)\sin\left(\frac{3\pi - 7\pi}{2}\right) \quad (2)$$

$$\sin\frac{3\pi}{11} - \sin\frac{7\pi}{11} = 2\cos\left(\frac{5\pi}{11}\right)\sin\left(-\frac{2\pi}{11}\right) = -2\cos\frac{5\pi}{11}\sin\frac{2\pi}{11}$$

$$= \frac{\sin(3x-x)}{\sin x \cos x} = \frac{\sin 2x}{\sin x \cos x} = \frac{2\sin x \cos x}{\sin x \cos x} = 2$$

تمرين 9: علما أن $\tan\left(\frac{x}{2}\right) = 3$ أحسب $\cos x$ و $\sin x$ و $\tan x$

الجواب: نستعمل القواعد: $\tan x = \frac{2t}{1-t^2}$ و $\cos x = \frac{1-t^2}{1+t^2}$ و $\sin x = \frac{2t}{1+t^2}$

$$\sin x = \frac{2 \times \tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} = \frac{2 \times 3}{1 + 3^2} = \frac{6}{10} = \frac{3}{5} \quad \text{و} \quad \tan x = \frac{2 \times \tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)} = \frac{2 \times 3}{1 - 3^2} = \frac{6}{-8} = -\frac{3}{4}$$

$$\sin x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} = \frac{1 - 3^2}{1 + 3^2} = \frac{-8}{10} = -\frac{4}{5}$$

تمرين 10: بين أن $\forall x \in \mathbb{R}$

$$\sin^2 2x - \cos 2x - 1 = -2\cos^2 x \times \cos 2x \quad (1)$$

$$2\sin^2 x + 12\cos^2 x = 5\cos 2x + 7 \quad (2)$$

أجوبة (1): $\sin^2 2x - \cos 2x - 1 = (2\cos x \sin x)^2 - 2\cos^2 x + 1 - 1 = 4\cos^2 x \sin^2 x - 2\cos^2 x = -2\cos^2 x \cos 2x$

$$4\cos^2 x \sin^2 x - 2\cos^2 x = -2\cos^2 x \cos 2x$$

$$2\sin^2 x + 12\cos^2 x = 2\sin^2 x + 12(1 - \sin^2 x) = -10\sin^2 x + 12 \quad (2)$$

$$= \frac{-10}{2}(1 - \cos 2x) + 12 = -5(1 - \cos 2x) + 12 = 5\cos 2x + 7$$

تمرين 11: بين أن $\forall x \in \mathbb{R}$

$$\sin 3x = \sin x \times (3 - 4\sin^2 x) \quad (1)$$

$$\cos 3x = \cos x(4\cos^2 x - 3) \quad (2)$$

$$\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1 \quad (3)$$

$$\sin(4x) = 4\sin x(2\cos^2 x - \cos x) \quad (4)$$

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x) \quad (5)$$

أجوبة (1): $\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$

$$= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x = 2\sin x(1 - \sin^2 x) + (1 - 2\sin^2 x)\sin x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x = \sin x(3 - 4\sin^2 x)$$

$$\cos 3x = \cos(2x+x) = \cos x \cos 2x - \sin 2x \sin x \quad (2)$$

$$= \cos x(2\cos^2 x - 1) + \sin x \times 2\cos x \sin x = 2\cos^3 x - \cos x - 2\cos x \sin^2 x$$

$$= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) = 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x = \cos x(4\cos^2 x - 3)$$

$$\cos(4x) = \cos(2 \times 2x) = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1 \quad (3)$$

$$= 2(4\cos^4 x - 4\cos^2 x + 1) - 1 = 8\cos^4 x - 8\cos^2 x + 1$$

$$\sin(4x) = \sin(2 \times 2x) = 2\sin 2x \cos 2x = 2 \times 2\sin x \cos x(2\cos^2 x - 1) \quad (4)$$

$$\sin(4x) = 4\sin x \cos x(2\cos^2 x - 1) = 4\sin x(2\cos^3 x - \cos x)$$

$$\text{؟؟} \quad \cos^3 x = \frac{1}{4}(3\cos x + \cos 3x) \quad (5)$$

طريقة 1:

$$\frac{1}{4}(3\cos x + \cos 3x) = \frac{1}{4}(3\cos x + \cos(x+2x)) = \frac{1}{4}(3\cos x + \cos x \cos 2x - \sin x \sin 2x)$$

$$= \frac{1}{4}(3\cos x + \cos x(2\cos^2 x - 1) - 2\sin x \sin x \cos x)$$

$$2\cos\left(x-\frac{\pi}{6}\right)=\sqrt{3} \Leftrightarrow \sqrt{3}\cos x+\sin x=\sqrt{3}$$

$$\cos\left(x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}=\cos\left(\frac{\pi}{6}\right) \Leftrightarrow 2\cos\left(x-\frac{\pi}{6}\right)=\sqrt{3}$$

يعني: $x-\frac{\pi}{6}=-\frac{\pi}{6}+2k\pi$ أو $x-\frac{\pi}{6}=\frac{\pi}{6}+2k\pi$

يعني: $x=2k\pi$ أو $x=\frac{\pi}{3}+2k\pi$

ومنه: $S=\left\{0;\frac{\pi}{3};2\pi\right\}$

$$\frac{\sin\frac{3\pi}{11}+\sin\frac{7\pi}{11}}{\sin\frac{3\pi}{11}-\sin\frac{7\pi}{11}}=\frac{2\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{2\pi}{11}\right)}{-2\cos\left(\frac{5\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)} \quad (3)$$

$$=\frac{\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{2\pi}{11}\right)}{\cos\left(\frac{5\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)}=-\tan\left(\frac{5\pi}{11}\right)\times\frac{1}{\tan\left(\frac{2\pi}{11}\right)}=-\frac{\tan\left(\frac{5\pi}{11}\right)}{\tan\left(\frac{2\pi}{11}\right)}$$

تمرين 16: بين أن: $\frac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x} = \tan 3x \times \tan x$

الجواب: $\cos 2x - \cos 4x = -2\sin\left(\frac{2x+4x}{2}\right)\sin\left(\frac{2x-4x}{2}\right) = 2\sin(3x)\sin x$

$$\cos 2x + \cos 4x = -2\cos\left(\frac{2x+4x}{2}\right)\cos\left(\frac{2x-4x}{2}\right) = 2\cos 3x \cos x$$

ملاحظة: $\sin(-x) = -\sin x$ و $\cos(-x) = \cos x$

$$\frac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x} = \frac{2\sin 3x \sin x}{2\cos 3x \cos x} = \frac{\sin 3x}{\cos 3x} \times \frac{\sin x}{\cos x} = \tan 3x \times \tan x$$

تمرين 17: بين أن: $\cos^2\frac{5x}{2} - \cos^2\frac{3x}{2} = -\sin 4x \times \sin x$

الجواب: $\cos^2\frac{5x}{2} - \cos^2\frac{3x}{2} = \left(\cos\frac{5x}{2} + \cos\frac{3x}{2}\right)\left(\cos\frac{5x}{2} - \cos\frac{3x}{2}\right)$

$$\cos\frac{5x}{2} + \cos\frac{3x}{2} = 2\cos\left(\frac{\frac{5x}{2} + \frac{3x}{2}}{2}\right)\cos\left(\frac{\frac{5x}{2} - \frac{3x}{2}}{2}\right) = 2\cos(2x)\cos\left(\frac{x}{2}\right)$$

$$\cos\frac{5x}{2} - \cos\frac{3x}{2} = -2\sin\left(\frac{\frac{5x}{2} + \frac{3x}{2}}{2}\right)\sin\left(\frac{\frac{5x}{2} - \frac{3x}{2}}{2}\right) = -2\sin(2x)\sin\left(\frac{x}{2}\right)$$

ومنه: $\cos^2\frac{5x}{2} - \cos^2\frac{3x}{2} = 2\cos(2x)\cos\left(\frac{x}{2}\right) \times -2\sin(2x)\sin\left(\frac{x}{2}\right)$

$$= -2\cos(2x) \times \sin(2x) 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) = -\sin(4x)\sin x$$

تمرين 18: بين أن: $\sin x + \sin 2x + \sin 3x = 2\sin x \cos x (1 + 2\cos x)$

الجواب:

$$\sin x + \sin 2x + \sin 3x = \sin 2x + \sin x + \sin 3x = \sin 2x + 2\sin 2x \cos x = \sin 2x + 2\sin 2x \cos x = \sin 2x(1 + 2\cos x) = 2\sin x \cos x (1 + 2\cos x)$$

تمرين 19: بين أن $\cos x - \sin x = \sqrt{2} \cos\left(\frac{\pi}{4} + x\right)$

الجواب: $a=1$ و $a=-1$

نحسب: $\sqrt{a^2 + b^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$$\cos x - \sin x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right) = \sqrt{2}\left(\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x\right)$$

$$\cos x - \sin x = \sqrt{2} \cos\left(\frac{\pi}{4} + x\right)$$

تمرين 20: حل في $[0; 2\pi]$ المعادلة: $\sqrt{3}\cos x + \sin x = \sqrt{3}$

الجواب: نحول أولا: $\sqrt{3}\cos x + \sin x$

$a=1$ و $a=\sqrt{3}$

نحسب: $\sqrt{a^2 + b^2} = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$

$$\sqrt{3}\cos x + \sin x = 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) = 2\left(\cos\frac{\pi}{6}\cos x + \sin\frac{\pi}{6}\sin x\right)$$

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$$